## LAMINAR FLOW IN CIRCULAR PIPE

The velocity distribution for Newtonian fluids can be obtained through the definition of viscosity. Equation (7.5) can be rewritten as

$$\mu = -\frac{\tau}{du/dr} \qquad \text{... (7.93)}$$

The negative sign in the above equation is incorporated to take into account the fact that in a pipe, u (velocity) decreases as r (radius) increases.

Rearranging Equation (7.93), we get

$$\frac{du}{dr} = -\frac{\tau}{\mu} \qquad \text{... (7.94)}$$

Substituting the value of $\tau$ from Equation (7.77) into Equation (7.94) gives

$$\frac{du}{dr} = -\frac{\tau_w}{r_w \, \mu} \cdot r \qquad \text{... (7.95)}$$

$$du = -\frac{\tau_w}{r_w \cdot \mu} \cdot r \, dr \qquad \text{... (7.96)}$$

Integrating Equation (7.96) with the boundary condition : At $r = r_w : u = 0$, we get

$$\int_0^u du = -\frac{\tau_w}{r_w \mu} \cdot \int_{r_w}^{r} r.dr \qquad \text{... (7.97)}$$

$$u = \frac{\tau_w}{2 r_w \mu} \left[r_w^2 - r^2\right] \qquad \text{... (7.98)}$$

The maximum value of the local velocity ($u_{max}$) is located at the centre of the pipe. At the centre of the pipe, $r = 0$ and $u = u_{max}$. Thus, from Equation (7.98), we get

$$u_{max} = \frac{\tau_w . r_w}{2 \mu} \qquad \text{... (7.99)}$$

Substituting for $\tau_w$ as $\tau_w = \dfrac{\Delta P.r_w}{2 \Delta L}$ into Equation (7.99) yields

$$u_{max} = \frac{\tau_w r_w}{2 \mu} \qquad \text{... (7.100)}$$

$$u_{max} = \frac{\Delta P . r_w^2}{4\mu \Delta L} \qquad \text{... (7.101)}$$

$$u_{max} = \frac{\Delta P . D^2}{16 \mu.\Delta L} \qquad [\text{as } D = 2 r_w] \qquad \text{... (7.102)}$$

Dividing Equation (7.98) by (7.100), we get

$$\frac{u}{u_{max}} = \left[1 - \left(\frac{r}{r_w}\right)^2\right]$$

or

$$u = u_{max}\left[1 - \left(\frac{r}{r_w}\right)^2\right] \qquad \text{... (7.103)}$$

It is clear from the form of Equation (7.103) that for laminar flow of Newtonian fluids through a circular pipe, the velocity distribution/profile with respect to radius is a parabola with the apex at the centreline of the pipe. Equation (7.103) relates the local velocity to the maximum velocity.

The average velocity u of the entire stream flowing through any given cross-sectional area (A) is defined by

$$u = \frac{1}{A} \int_A u.dA \qquad \text{... (7.104)}$$

Substituting the values of A, u and dA into Equation (7.104), we get

$$u = \frac{\tau}{r_w^3 \mu} \int_0^{r_w} \left(r_w^2 - r^2\right) r.dr \qquad \text{... (7.105)}$$

$$u = \frac{\tau_w r_w}{4 \mu} \qquad \text{... (7.106)}$$

[As $A = \pi r_w^2$ and $dA = 2 \pi r\, dr$ = area of elementary ring of radius r and width dr].

From Equations (7.100) and (7.106), we get

$$\frac{u}{u_{max}} = 0.5 \qquad \ldots (7.107)$$

i.e., *the average velocity is one half the maximum velocity.*

Eliminating $\tau_w$ by replacing it by $\Delta P$, using $\tau_w = \frac{\Delta P \, r_w}{2 \, \Delta L}$, from Equation (7.106) and replacing $r_w$ by D/2, we get

$$u = \frac{\Delta P \, D^2}{32 \, \Delta L \, \mu} \qquad \ldots (7.108)$$

Rearranging Equation (7.108), we get

$$\Delta P = \frac{32 \, \Delta L \, \mu \, u}{D^2} \qquad \ldots (7.109)$$

$\Delta L$ can be replaced by L.

$$\therefore \qquad \Delta P = \frac{32 \, L \, \mu \, u}{D^2} \qquad \ldots (7.109a)$$

where $\Delta P$ is the pressure drop over the length L of the pipe.

Equation (7.109) is the Hagen-Poiseuille equation. The Hagen-Poiseuille equation is useful to determine experimentally the viscosity of a fluid by measuring the pressure drop and the volumetric flow rate through a tube of a given length and diameter. This equation also useful for the calculation of the pressure drop due to friction in laminar/viscous flow if the viscosity is known.

The head loss due to friction in laminar flow is given by

$$h'_{fs} = h_{fs}/g = \frac{\Delta P}{\rho \, g} = \frac{32 \, \mu \, L \, u}{\rho \, g \, D^2} \qquad \ldots (7.110)$$

We know that,

$$u = \frac{\tau_w \cdot r_w}{4 \, \mu}$$

$$u = \frac{\tau_w . D}{8 \, \mu} \qquad [\text{as } D = 2 \, r_w] \qquad \ldots (7.111)$$

Rearranging Equation (7.111), we get

$$\tau_w = \frac{8 \, u . \mu}{D} \qquad \ldots (7.112)$$

Substituting the value of $\tau_w$ from Equation (7.112) into Equation (7.84), we get

$$f = \frac{2}{\rho u^2}\left(\frac{8 \, u \, \mu}{D}\right) \qquad \ldots (7.113)$$

$$f = 16\left(\frac{\mu}{D \, \rho \, u}\right) \qquad \ldots (7.114)$$

$$f = \frac{16}{\left(\frac{Du\rho}{\mu}\right)} \qquad \ldots (7.115)$$

$$f = \frac{16}{N_{Re}} \qquad \ldots (7.116)$$

Thus, for laminar flow through pipes, the Fanning friction factor is calculated with the help of Equation (7.116) knowing $N_{Re}$ (Reynolds number).