Circular Convolution

This property states that if $x_1(n) \longleftrightarrow X_1(k)$

$x_2(n) \longleftrightarrow X_2(k)$

$$x_1(n) \, (N) \, x_2(n) \longleftrightarrow X_1(k) X_2(k).$$

$x_1(n) \, (N) \, x_2(n)$ means circular convolution of $x_1(n)$ and $x_2(n)$.
This property states that multiplication of two DFT's is equivalent to circular convolution of their sequence in time domain.

$\Rightarrow$

By definition two DFT's $X_1(k)$ and $X_2(k)$.

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n) e^{-j\frac{2\pi kn}{N}} \quad \text{--- (1)}$$

$$X_2(k) = \sum_{l=0}^{N-1} x_2(l) e^{-j\frac{2\pi kl}{N}} \quad \text{--- (2)}$$

Now $X_3(k) = X_1(k) * X_2(k)$ --- (3)

Let $x_3(m)$ be the sequence whose DFT is $X_3(k)$

$$x_3(m) = \frac{1}{N} \sum_{k=0}^{N-1} X_3(k) e^{j\frac{2\pi km}{N}} \quad \text{--- (4)}$$

Sub $X_3(k)$ in (4)

$$x_3(m) = \frac{1}{N} \sum_{k=0}^{N-1} [X_1(k) * X_2(k)] e^{j\frac{2\pi km}{N}} \quad \text{--- (5)}$$

Sub the value of $X_1(k)$ and $X_2(k)$ in (5).

$$x_3(m) = \frac{1}{N} \sum_{k=0}^{N-1} \left[\sum_{n=0}^{N-1} x_1(n) e^{-j\frac{2\pi kn}{N}} \right] \left[\sum_{l=0}^{N-1} x_2(l) e^{-j\frac{2\pi kl}{N}} \right] e^{j\frac{2\pi km}{N}}.$$

All 3 summations have different indices rearranging summation.

$$x_3(m) = \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \left[\sum_{k=0}^{N-1} e^{j\frac{2\pi k(m-n-l)}{N}} \right] \quad \text{--- (6)}$$

From geometric series formula.

$$\sum_{K=0}^{N-1} a^K = \begin{cases} N & \text{for } a=1 \\ \frac{1-a^N}{1-a} & \text{for } a\neq 1 \end{cases} \quad \text{—— (7)}$$

Let $a = e^{\frac{j2\pi K(m-n-l)}{N}}$

when $(m-n-l) = N, 2N, 3N \ldots$ ie multiple of N ; $a=1$

$\therefore a^{\frac{j2\pi N}{N}} = e^{\frac{j2\pi 2N}{N}} = e^{\frac{j2\pi 3N}{N}} \ldots\ldots = 1$

So Equation (7) Can be written as.

(1) Condition

$$\sum_{K=0}^{N-1} a^K = \sum_{K=0}^{N-1} e^{j2\pi K(m-n-l)/N} = N.$$

(2) Condition $a \neq 1$

$$\sum_{K=0}^{N-1} a^K = \sum_{K=0}^{N-1} e^{j2\pi K(m-n-l)/N}$$

$$= \frac{1-a^N}{1-a} = \frac{1 - e^{\frac{j2\pi(m-n-l)}{\cancel{N}}\times \cancel{N}}}{1-e^{j2\pi(m-n-l)/N}}$$

Hence $e^{j2\pi(m-n-l)} = 1$

$$= \frac{1-1}{1-e^{j2\pi(m-n-l)/N}}$$

$$= 0$$

$$\sum_{K=0}^{N-1} e^{j2\pi K(m-n-l)/N} = \begin{cases} N & \text{when } (m-n-l) \text{ is multiple of } N \\ 0 & \text{otherwise.} \end{cases}$$

Sub ~~Eq~~ above Condition in Eq (6)

$$x_3(m) = \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) . N$$

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l)$$ | when $m-n-l$ is multiple of N.

$\therefore$ $m - n - l = pN$ p is some integer, either +ve or -ve.

$$l = m - n + pN$$

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(m-n+pN)$$

So $x_2(m-n+pN) = x_2(m-n, \bmod N)$

$$= x_2((m-n))_N$$

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n)\, x_2((m-n))_N .$$

$\therefore$ Above equation can be written as

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

Circular time shift of sequence

if $x(n) \longrightarrow X(k)$

$x((n-l))_N \longleftrightarrow X(k)\, e^{-j\frac{2\pi k l}{N}}$

$\Longrightarrow$

$$DFT\{x(n)\} = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi k n}{N}}$$

$$\{x((n-l))_N\} = \sum_{n=0}^{N-1} x((n-l))_N\, e^{-j\frac{2\pi k n}{N}}$$

$$= \sum_{n=0}^{l-1} x((n-l))_N\, e^{-j\frac{2\pi k n}{N}} + \sum_{n=l}^{N-1} x((n-l))_N\, e^{-j\frac{2\pi k n}{N}}$$

But $x((n-l))_N = x(N-l+n)$

$$\sum_{n=0}^{l-1} x((n-l))\, e^{-j\frac{2\pi k n}{N}} = \sum_{n=0}^{l-1} x(N-l+n)\, e^{-j\frac{2\pi k n}{N}}$$

$$= \sum_{m=N-l}^{N-1} x(m)\, e^{-j2\pi k(m+l)/N}$$

$$= X(k)\, e^{-j\frac{2\pi k l}{N}}$$

Compute the 5-point DFT of the sequence $x(n) = (1,0,1,0,1)$ and hence verify the symmetry property.

Solution

$$DFT[x(n)] = X(k)$$

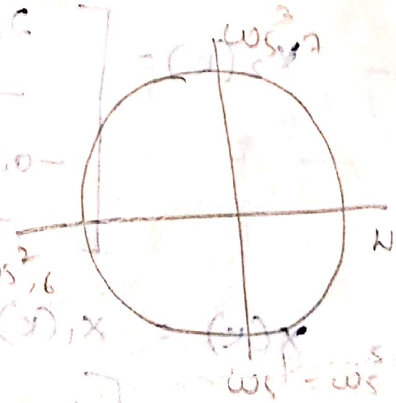
$$= \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \qquad k = 0, 1 \ldots N-1$$

For $N=5$

$$X(k) = \sum_{n=0}^{4} x(5)\, W_5^{kn}$$

$$W_5 = \begin{bmatrix} W_5^0 & W_5^0 & W_5^0 & W_5^0 & W_5^0 \\ W_5^0 & W_5^1 & W_5^2 & W_5^3 & W_5^4 \\ W_5^0 & W_5^2 & W_5^4 & W_5^6 & W_5^8 \\ W_5^0 & W_5^3 & W_5^6 & W_5^8 & W_5^{12} \\ W_5^0 & W_5^4 & W_5^8 & W_5^{12} & W_5^{16} \end{bmatrix} \qquad x(5) = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$X(k) = \begin{bmatrix} W_5^0 + W_5^0 + W_5^0 \\ W_5^0 + W_5^2 + W_5^4 \\ W_5^0 + W_5^4 + W_5^8 \\ W_5^0 + W_5^6 + W_5^{12} \\ W_5^0 + W_5^8 + W_5^{16} \end{bmatrix}$$



$$W_5^0 = e^{-j\frac{2\pi}{N}} = e^{-j\frac{2\pi}{5}\times(0)} = e^{1} = 1$$

$$W_5^1 = e^{-j\frac{2\pi}{5}\times 1} = \cos\frac{2\pi}{5} - j\sin\frac{2\pi}{5} = 0.309 - j0.951 = W_5^{11} = W_5^{16}$$

$$W_5^2 = e^{-j\frac{2\pi}{5}\times 2} = \cos\frac{4\pi}{5} - j\sin 4\pi/5 = -0.809 - j0.587 = W_5^{12} = W_5^{17}$$

$$W_5^3 = e^{-j\frac{2\pi}{5}\times 3} = \cos\frac{6\pi}{5} - j\sin\frac{6\pi}{5} = -0.809 + j0.587 = W_5^8 = W_5^{13}$$

$W_5^9 = W_5^4 = W_5^4 = e^{-j\frac{2\pi}{5}\times 4} = \cos\frac{8\pi}{5} - j\sin\frac{8\pi}{5} = 0.309 + j0.951$

$$x(k) = \begin{bmatrix} 1 & + & 1 & + & 1 \\ 1 & + 0.809 - j0.587 & & + 0.309 + j0.951 & \\ 1 & + 0.309 + j0.951 & & - 0.809 + j0.587 & \\ 1 & + 0.309 - j0.951 & & - 0.809 - j0.587 & \\ 1 & - 0.809 + j0.587 & & + 0.309 - j0.951 & \end{bmatrix}$$

$$X(k) = \begin{bmatrix} 3 \\ 0.5 + j0.364 \\ 0.5 + j1.538 \\ 0.5 - j1.538 \\ 0.5 - j0.364 \end{bmatrix} = \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \end{bmatrix}$$

Symm property shows that $X(k) = X_R(k) + jX_I(k)$

The first five points of the 8 point DFT of a real valued sequence are $(0.25, (0.5 - j0.5), 0, (0.5 - j0.86), 0)$. Find the remaining 3 points.

<u>Solution</u> From the property of Conjugate Symmetry.

$X(k) = X^*(N-k)$ $\boxed{N=8}$ $k = 0, 1 \ldots 7$

$X(5) = X^*(8-5)$

$X(5) = X^*(3)$

$X(6) = X^*(2)$ $\quad X^*(8-6)$

$X(7) = X^*(1)$ $\quad X^*(8-7)$

Given:- $X(0) = 0.25$

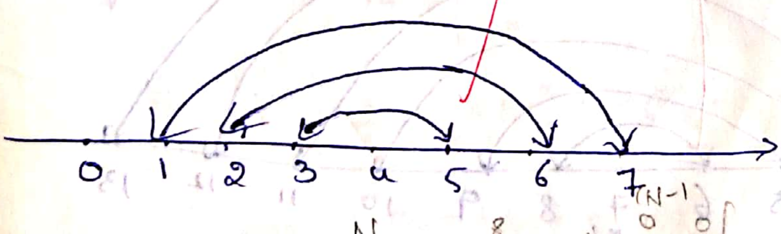
$X(1) = 0.5 - j0.5$

$X(3) = 0.5 - j0.86$

$X(2) = 0$

$X(4) = 0$

If N is odd then Conjugate Symm is about $\frac{N}{2}$ which is Called folding Index



$\frac{N}{2} = \frac{8}{2} = 4$ is mid point

The complete 8-point sequence, $X(k)$ is tabulated

K	$X(K)$	K	$X^*(k)$
0	0.25	4	0
1	$0.5-j0.5$	7	$0.5+j0.5$
2	0	6	0
3	$0.5-j0.86$	5	$0.5+j0.86$

So symmetry prop. has been used

Let $X(K)$ be a 14-point DFT of length-14 real sequence $x(n)$. The first 8 samples of $X(k)$ are given by

$X(0)=12$ $\quad X(1)=-1+j3$ $\quad X(2)=3+j4$ $\quad X(6)=-2-3j$

$X(3)=1-j5$ $\quad X(4)=-2+j2$ $\quad X(5)=6+j3$ $\quad X(7)=10$

Find the remaining samples of $X(k)$. Also Evaluate the following (a) $x(0)$ b) $x(7)$ c) $\sum_{n=0}^{13} x(n)$

d) $\sum_{n=0}^{13} |x(n)|^2$.

Solution

Since $x(n)$ is a real sequence, the following symmetry condition must be satisfied.

$$X(k) = X^*(N-k), \quad 0 \le K \le N-1$$

Since $N=14$ $\quad X(K) = X^*(14-K) \quad 0 \le K \le 13.$

$\frac{N}{2} = \frac{14}{2} = 7$

7 is mid point.

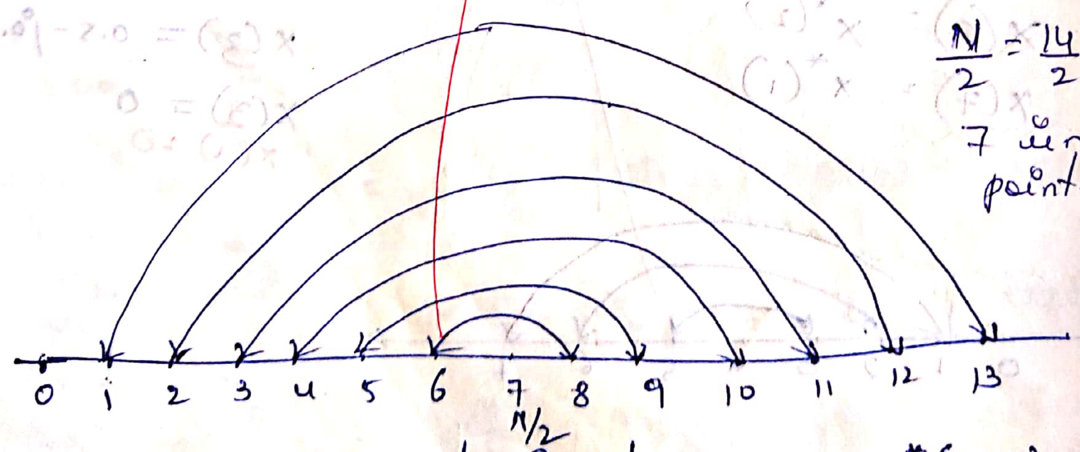


Fig1: Conjugate Symmetry : $X(k) = X^*(14-k)$

$$X(8) = X^*(6) = -2 + j3 \qquad X(6) = -2 - 3j$$

$$X(9) = X^*(5) = 6 - j3 \qquad X(5) = 6 + j3$$

$$X(10) = X^*(4) = -2 - 2j \qquad X(4) = -2 + j2$$

$$X(11) = X^*(3) = 1 + j5 \qquad X(3) = 1 - j5$$

$$X(12) = X^*(2) = 3 - j4 \qquad X(2) = 3 + j4$$

$$X(13) = X^*(1) = -1 - j3. \qquad X(1) = -1 + j3$$

The result is tabulated below

K	X(K)	K	X*(K)
0	12	7	10
1	-1+j3	8	-2+j3
2	3+j4	9	6-j3
3	1-j5	10	-2-j2
4	-2+j2	11	1+5j
5	6+j3	12	3-j4
6	-2-j3	13	-1-j3

a. From definit of N-point Inverse DFT we have

$$x(n) = \frac{1}{N} \sum_{n=0}^{N-1} X(k)\, W_N^{-kn} \qquad 0 \le K \le N-1$$

$$x(n) = \frac{1}{14} \sum_{K=0}^{14-1} X(k)\, W_{14}^{-kn} \qquad 0 \le K \le 14-1$$

Let $n=0$ on both side of the above Equation

$$x(0) = \frac{1}{14} \sum_{k=0}^{13} X(k)$$

$$= \frac{1}{14}\left[X(0) + X(1) + \dots + X(13)\right]$$

$$= \frac{1}{14}\left[12 - 1 + j3 + 3 + j4 + 1 - j5 - 2 + j5 + 6 + j3 - 2 - j3 + 10 - 2 + j3 + 6 - j3 - 2 - j2 + 1 + 5j + 3 - j4 - 1 - j3\right]$$

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$= 2.2857$

b) $x(7)$ From fig (1) $x(7) = \frac{N}{2}$ $\quad n = \frac{N}{2}$

$$x\left(\frac{N}{2}\right) = \frac{1}{N}\sum_{K=0}^{N-1} X(K)\, W_N^{-K\frac{N}{2}}$$

W_N^{KT} ... $W_N^{Kn} = e^{-\frac{j2\pi Kn}{N}}$

$W_N^{-Kn} = e^{\frac{j2\pi Kn}{N}} = e^{\frac{j2\pi K}{N}\cdot\frac{N}{2}} = e^{j\pi K}$

$x\left(\frac{N}{2}\right) = 12 - (-1+3j) + (3+j4) - (1-j5) + (-2+2j) - (6+j3) + (-2-3j) - 10 + (-2+3j) - (6-3j) + (-2-2j) - (1+j5) + (3-j4) - (-1-3j)$

$= \frac{-12}{14} = -0.8757$

$$= \frac{1}{N}\sum_{K=0}^{N-1} X(K)\, e^{-j\frac{2\pi}{N}\cdot -K\frac{N}{2}}$$

$$= \frac{1}{N}\sum_{K=0}^{N-1} X(K)\, e^{j\frac{2\pi K}{N}\cdot\frac{N}{2}}$$

$$= \frac{1}{N}\sum_{K=0}^{N-1} X(K)\, e^{j\pi K} \qquad \cos\pi + j\sin\pi = (-1)^K$$

$$= \frac{1}{N}\sum_{K=0}^{N-1} X(K)\,(-1)^K$$

$N = 14$

$$= \frac{1}{14}\sum_{n=0}^{14-1} X(K)\,(-1)^K$$

$x\left(\frac{N}{2}\right) = \frac{1}{14}\left[X(0)(-1)^0 + X(1)(-1)^1 + X(2)(-1)^2 + X(3)(-1)^3 + X(4)(-1)^4 + X(5)(-1)^5 + X(6)(-1)^6 + X(7)(-1)^7 + X(8)(-1)^8 + X(9)(-1)^9 + X(10)(-1)^{10} + X(10)(-1)^{11} + X(11)(-1)^{12} + X(12)(-1)^{13}\right]$

$= -0.8571$

$= \frac{1}{14}\left[X(0) - X(1) + X(2) - X(3) + X(4) - X(5) + X(6) - X(7) + X(8) - X(9) + X(10) - X(11) + X(12) - X(13)\right]$

c) By definition

$$\text{DFT}\{x(n)\} = X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{Kn} \qquad 0 \le K \le N-1$$

$$X(K) = \sum_{n=0}^{14-1} x(n)\, W_{14}^{Kn}$$

Let $K=0$ on BS

$$X(0) = \sum_{n=0}^{13} x(n)\, W_{14}^{0}$$

$$X(0) = \sum_{n=0}^{13} x(n) = 12$$

d. From Parseval's theorem.

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{n=0}^{N-1} |X(k)|^2$$

Not in syllabus

$$\sum_{n=0}^{14-1} |x(n)|^2 = \frac{1}{14} \sum_{n=0}^{13-1} |X(k)|^2$$

$$= \frac{1}{14}\Big(144+10+25+26+8+45+13+100+13+45+8+26+25+10\Big)$$

$$= \underline{\underline{35.5714}}$$

Let $x(n) = (1, 2, 0, 3, -2, 4, 7, 5)$ Evaluate the following a) $X(0)$ b) $X(4)$ (c) $\sum_{K=0}^{7} x(k)$

a. By definition

$$DFT\{x(n)\} = X(k)$$

$$= \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \qquad 0 \le k \le N-1$$

Let $k=0$

$N=0$

$$X(0) = \sum_{n=0}^{8-1} x(n)\, W_N^{0} \qquad W_N^0 = 1$$

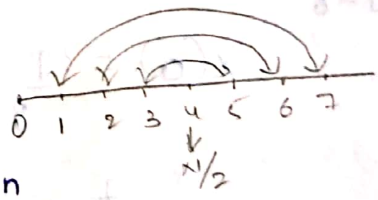
$N=8$

$k = \frac{N}{2} = \frac{8}{2} = 4$

$k=4$ MP

$$= 1+2+0+3-2+4+7+5$$

$$\boxed{X(0) = 20}$$



b) Letting $k = \frac{N}{2}$.

$$X\left(\frac{N}{2}\right) = \sum_{n=0}^{8-1} x(n)\, W_N^{\frac{N}{2}n}$$

$$= \sum_{n=0}^{8-1} x(n)\, e^{-j\frac{2\pi}{N} \times \frac{N}{2} n}$$

$$= \sum_{n=0}^{7} x(n)\, e^{-j\pi n}$$

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$$= \sum_{n=0}^{N-1} x(n)\,(-1)^n$$

Here N=8

$$X\left(\frac{8}{2}\right) = \sum_{n=0}^{8-1} x(n)\,(-1)^n$$

$$X(4) = \sum_{n=0}^{7} x(n)\,(-1)^n$$

$x(0)(-1)^0 + x(1)(-1)^1 + x(2)(-1)^2 + x(3)(-1)^3 + x(4)(-1)^4 + x_5(-1)^5 + x_6(-1)^6 + x_7(-1)^7 =$

$$= x(0) - x(1) + x(2) - x(3) + x(4) - x(5) + x(6) - x(7)$$

$$= 1-2+0-3-2-4+7-5$$

$$= -8$$

c. From the definition of inverse DFT we have

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-kn}$$

n=0 on Both side

$$x(0) = \frac{1}{N} \sum_{k=0}^{N-1} X(k).$$

N=8

$$x(0) = \frac{1}{8} \sum_{k=0}^{8-1} X(k)$$

$$\sum_{k=0}^{7} X(k) = 8\, x(0) \text{ (Given)}$$

$$= 8 \times 1$$

$$= 8.$$