Chapter 4

Maxwell's Equations

4.1 Fundamentals of vector calculus

4.1.1 Dot product or Scalar product

The dot product of two vectors is defined as follows

$$\vec{a}.\vec{b} = abcos\theta \tag{4.1}$$

here θ is the angle between two vectors. a and b are the magnitudes of $\vec{a}$ and $\vec{b}$. If $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{z}$ and $\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{z}$ then the dot product or scalar product is given by

$$\vec{a}.\vec{b} = a_x b_x + a_y b_y + a_z b_z \tag{4.2}$$

The dot product of two vectors is a scalar quantity.

Physical Significance The dot product is mathematically put forward and could be applied in physics under suitable circumstances. For example the work done is maximum when the displacement is along the force. Thus work done is defined as the dot product of force ($\vec{F}$) and displacement($\vec{d}$) and is a scalar quantity. Hence $W = \vec{F}.\vec{d}$.

4.1.2 Vector product or Cross product

The vector product of two vectors is defined as follows

$$\vec{a} \times \vec{b} = a\ b\ sin\theta\ \hat{n} \tag{4.3}$$

here θ is the angle between two vectors. a and b are the magnitudes of $\vec{a}$ and $\vec{b}$. $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$. If $\vec{a} = a_x\hat{i}+a_y\hat{j}+a_z\hat{z}$ and $\vec{b} = b_x\hat{i}+b_y\hat{j}+b_z\hat{z}$ then their cross product is given by

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \tag{4.4}$$

The cross product of two vectors is a vector quantity.

Physical Significance The cross product is put forward in mathematics an could be applied in physics under suitable circumstances. For a rotating body the moment of linear momentum is the angular momentum. The angular momentum acts in a direction perpendicular to momentum and the radius vector. Thus angular momentum ($\vec{L}$) is given by the cross product of radius vector ($\vec{r}$) and linear momentum ($\vec{p}$) and hence $\vec{L} = \vec{r} \times \vec{p}$.

4.1.3 Scalar field

It is a function of a space whose value at each point is a scalar quantity. For example potential setup by a charge in space.

4.1.4 Vector field

It is a function of a space whose value at each point is a vector quantity. Consider a region in the flowing water. Each and every point can be associated with a vector whose magnitude represents the speed of flow and direction gives the direction of flow. Thus the whole region could be imagined filled with vectors and is an example of vector field. Consider a region surrounding a point charge. The electric field at each and every point surrounding the charge could be represented by vectors and hence is a vector field.

4.1.5 The ∇ Operator

In mathematics the following operator is used called ∇ operator. When this operator acts on a scalar quantity it instructs to differentiate the scalar quantity. The operation of ∇ on a scalar quantity results in a vector quantity. The ∇ operator is given by

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \tag{4.5}$$

Let T be a scalar function. Then ∇T states that the ∇ acts on T. There are three ways in which ∇ can act.

1. On a scalar function ∇T called the **Gradient**.
2. On a vector function via the dot product $\nabla.\vec{A}$ called the **Divergence**.
3. On a vector function via the cross product $\nabla\times\vec{A}$ called the **Curl**.

4.1.6 The Gradient

Consider a scalar function V. The operator ∇ acting on the scalar function V is given by

$$\nabla V = \hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z} \tag{4.6}$$

The gradient ∇V points along the maximum variation of the function V and the magnitude of ∇V gives the rate of change in the maximal direction.

Physical significance: Let us consider a positive point charge in space. Let the potential set up by the charge in the surrounding be V and is a scalar quantity. The potential decreases as the distance from the charge increases. Thus the gradient of potential results in the electric field strength which is a vector quantity. This could be written as

$$\vec{E} = -\frac{\partial V}{\partial r}\hat{r} \tag{4.7}$$

Here r is the position vector and $\hat{r}$ is the unit vector along position vector. The negative sign indicates the decrease in potential. Thus the above equation could be written as

$$\vec{E} = -\nabla V = \frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k} \tag{4.8}$$

Thus the Electric field strength is defined as negative of gradient of potential also known as $grad\ V$.

4.1.7 The Divergence

The divergence of a vector field is mathematically written as $\nabla.\vec{E}$. The vector field E is represented by $\vec{E} = E_x\hat{i} + E_y\hat{j} + E_z\hat{k}$. From the definition of the ∇ we can construct divergence as

$$\nabla.\vec{E} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(E_x\hat{i} + E_y\hat{j} + E_z\hat{k}\right)$$

$$\nabla.\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \tag{4.9}$$

From the equation 4.9 we observe that the divergence of a vector field is a scalar quantity.

Physical significance : The physical significance of the divergence of a vector function is it measures how much the vector E spreads out (diverges) from a point of consideration. For example if we consider a positive charge in space the field lines diverge and hence it is **positive divergence**. For a negative charge the field lines converge and hence it is **negative divergence**. If the field lines or parallel then it is**zero divergence**. See fig. 4.1.

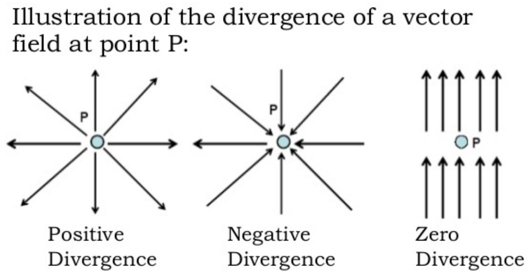

Figure 4.1: Positive, Negative and Zero Divergence

4.1.8 The Curl

The curl of a vector field is could be constructed as follows

$$\nabla \times \vec{H} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(H_x\hat{i} + H_y\hat{j} + H_z\hat{k}\right)$$

$$\nabla \times \vec{H} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} \tag{4.10}$$

The equation 4.10 represents curl of $\vec{H}$ and also it is evident that curl of a vector is a vector quantity.

Physial significance : The curl of a vector function is a measure how much field swirls (curls) around the point of consideration. Consider a wire carrying electric current. This sets magnetic field surrounding the wire. Consider a point on the wire. The magnetic field lines curl or swirl around the point. Higher the value of $\vec{H}$ around the point stronger will be the curl. If the field lines purely parallel then it represents zero curl around the point. See fig 4.2.

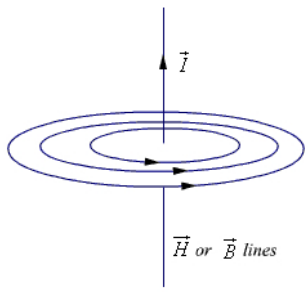

Figure 4.2: Curl of a magnetic field

4.2 Line, Surface and Volume integrals

4.2.1 Line integral

Line integral is an expression of the form

$$\int_{P}^{A} \vec{A}.\vec{dl} \tag{4.11}$$

here $\vec{A}$ represents the vector field and dl represents a in-

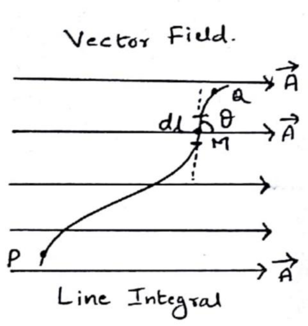

Figure 4.3: Line Integral

finitesimally small length at a point M along the path PQ in the field. The dot product $\vec{A}.vecdl$ and θ is the angle made dl with $\vec{A}$. For a closed path the integral is written as

$$\oint \vec{A}.\vec{dl} \tag{4.12}$$

$\oint$ is the symbol used for closed contour integral. This is also called as circulation of $\vec{A}$ around the closed path. The

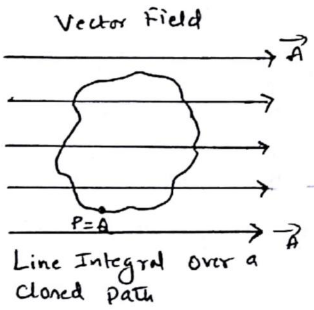

Figure 4.4: Circulation of vector

line integral concept can be applied to calculate the potential difference between to points in an electric field.

4.2.2 Surface integral

Consider a surface of area S in a vector field $\vec{A}$. consider a small infinitesimal area dS on the surface around point M as in the figure. Consider $\hat{n}$ a unit vector normal to ds and $dS\ \hat{n}$ represents area vector of $\vec{dS}$. The surface integral over the entire surface S is given by

$$\int_{s} \vec{A}.\vec{dS} \tag{4.13}$$

Here $\int_s$ is the symbol used for surface integral. The sur-

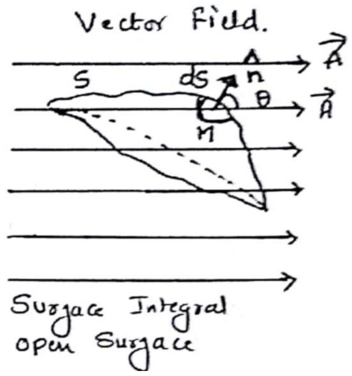

Figure 4.5: Surface Integral

face integral gives the net outward flux of the vector field through the surface. For a closed surface the surface integral is given by

$$\oint_{s} \vec{A}.\vec{dS} \tag{4.14}$$

In case of surface integral for a closed surface the $\hat{n}$ chosen outwards. The surface integral could be applied to calculate the net flux of the electric field through a surface in the electric field.

4.2.3 Volume integral

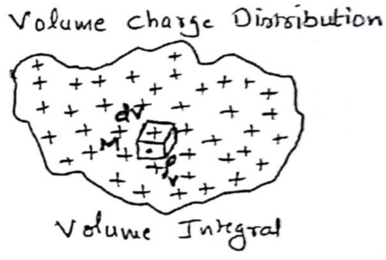

Figure 4.6: Volume Integral

Consider a volume charge distribution in which charges are continuously distributed. Let v be the volume through which the charges are distributed. Consider a point M inside the charge distribution. Let dv be a small volume around a point M. let ρ_v be the density of charges at M

and is a scalar quantity. The net charge in the volume is given by volume integral of the form

$$\oint_v \rho_v dv \tag{4.15}$$

here $\oint_v$ is the symbol for volume integral.

4.3 Some Theorems of Electrostatics, Electricity, Magnetism and Electromagnetic induction

4.3.1 Gauss flux theorem - Gauss' law in electrostatics

Consider a region in space consisting of charges. Let a surface of any shape enclose these charges and is called a Gaussian surface. Let q be the charge enclosed by a closed surface S. The closed surface could be considered to be made up of number of elementary surfaces dS. If $\vec{D}$ is the electric flux density at dS then the surface integral gives the total electric flux over the surface S could be obtained as

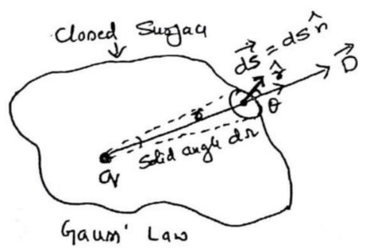

Figure 4.7: Gauss' Flux Theorem - Electrostatics

$$\phi = \oint_s \vec{D}.\vec{dS} = \sum q \tag{4.16}$$

here ϕ is the total flux and $\sum q = (q_1 + q_2 + ...)$ is the total charge enclosed by the surface.

4.3.2 Gauss Divergence Theorem

Divergence of $\vec{D}$

Consider a vector field $\vec{D}$. Consider a point P in the vector field. Let ρ_v be the density of charges at the point P. It can be shown that the divergence of the $\vec{D}$ is given by

$$\nabla.\vec{D} = \rho_v \tag{4.17}$$

This is also the Maxwell's first equation.

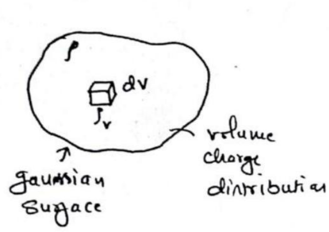

Figure 4.8: Gauss divergence theorem

Statement: The Gauss divergence theorem states that the integral of the normal component of the flux density over a closed surface of any shape in an electric field is equal to the volume integral of the divergence of the flux throughout the space enclosed by the Gaussian surface. Mathematically

$$\oint_s \vec{D}.\vec{dS} = \oint_v (\nabla.\vec{D})\ dv \tag{4.18}$$

Proof

Consider a volume v enclosed by a Gaussian surface S. Let a charge dQ be enclosed by a small volume dv inside the Gaussian surface. If ρ is the density of charges and may vary inside the volume v then the charge density associated with volume dv is given by

$$\rho_v = \frac{dQ}{dv}$$

Thus

$$dQ = \rho_v\ dv$$

Thus the total charge enclosed by the Gaussian surface is give by

$$Q = \oint_v dQ = \oint_v \rho_v\ dv$$

Substituting for ρ_v from Maxwell's First equation 4.18 we get

$$Q = \oint_v (\nabla.\vec{D})\ dv$$

According to Gauss' law of electrostatics we have

$$Q = \oint_s \vec{D}.\vec{dS}$$

Thus equating the equations for Q we get

$$\oint_s \vec{D}.\vec{dS} = \oint_v (\nabla.\vec{D})\ dv \tag{4.19}$$

Thus Gauss divergence theorem. Divergence theorem relates the surface integral with volume integral.

4.3.3 Stokes' Theorem

Stokes, theorem relates surface integral with line integral (Circulation of a vector field around a closed path).

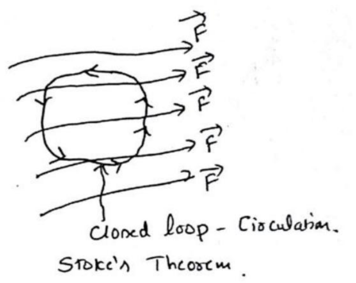

Figure 4.9: Stokes' theorem

Statement: The surface integral of curl of $\vec{F}$ throughout a chosen surface is equal to the circulation of the $\vec{F}$ around the boundary of the chosen surface.

Mathematically

$$\int_s (\nabla \times \vec{F}).\vec{dS} = \oint \vec{F}.\vec{dl} \qquad (4.20)$$

4.3.4 Gauss' law of Magnetostatics

Consider a closed Gaussian surface of any shape in a magnetic field. The magnetic fields lines exist in closed loops. Hence for every flux line that enters the closed surface a flux line emerges out else where. Thus for a closed surface in a magnetic field the total inward flux(Positive) is equal to total outward flux(Negative). Thus the net flux through the Gaussian surface is zero. Thus it could be written

$$\oint_s \vec{B}.\vec{dS} = 0 \qquad (4.21)$$

Here $\vec{B}$ magnetic flux density. Applying Gauss divergence theorem we get

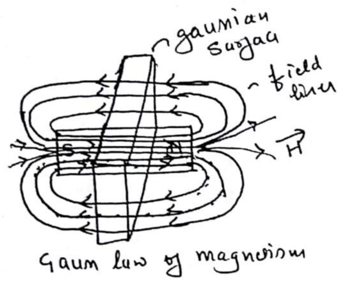

Figure 4.10: Gauss' Flux Theorem - Magnetostatics

$$\oint_s \vec{B}.\vec{dS} = \oint_v (\nabla.\vec{B})\ dv = 0$$

Hence it could be written

$$\nabla.\vec{B} = 0 \qquad (4.22)$$

This is one of the Maxwell's equations.

4.3.5 Amperes Law

Statement: The circulation of magnetic field strength $\vec{H}$ along a closed path is equal to the net current enclosed (I_{enc}) by the loop. Mathematically

$$\oint \vec{H}.\vec{dl} = I_{enc} \qquad (4.23)$$

By applying stokes' theorem we get

$$\int_s (\nabla \times \vec{H}).\vec{dS} = I_{enc} \qquad (4.24)$$

The equation for I_{enc} could be obtained as

$$I_{enc} = \oint_s \vec{J}.\vec{dS} \qquad (4.25)$$

Equating equations 4.24 and 4.25 we get

$$\int_s (\nabla \times \vec{H}).\vec{dS} = \oint_s \vec{J}.\vec{dS}$$

Thus we get the amperes law as

$$\nabla \times \vec{H} = \vec{J} \qquad (4.26)$$

Thus Amperes circuital law and another Maxwell's equation.

4.3.6 Biot-Savart Law

Considera a portion of a conductor carrying current I. Let dl be infinitesimally small elemental length of the conductor at M. Consider a point P near The conductor. Let $\vec{MP}$ be the the vector joining the element with the point and of length r with $\hat{r}$ being the unit vector. θ is the angle made by MP with the element. Biot-Savart law states the magnitude and direction of the small magnetic field at P due to the elemental length dl of the current carrying conductor.

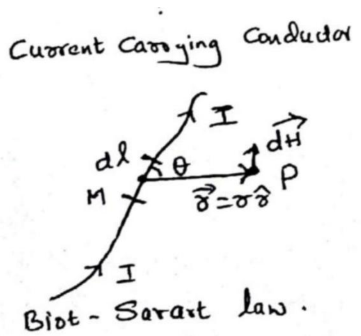

Figure 4.11: Biot-Savart Law

The magnitude of the magnetic field $\vec{dH}$ is

1. Proporitonal to the length of the element dl

2. Proportional to the current through the element I.
3. Proportional to the Sine of the angle θ, $Sin(\theta)$.
4. Inversely proportional to the square of the distance r.

The direction of the magnetic field $\vec{dH}$ is perpendicular to the plane containing both the element and the vector $\vec{r}$. Mathematically we get

$$dH \propto \frac{I\ dl\ Sin(\theta)}{r^2}$$

$$dH = \frac{I\ dl\ Sin(\theta)}{4\pi r^2} \tag{4.27}$$

Here $\frac{1}{4\pi}$ is the proportionality constant. the above equation could be expressed in the vector form as

$$\vec{dH} = \frac{I\ \vec{dl} \times \hat{r}}{4\pi r^2} \tag{4.28}$$

Thus the Biot-Savart Law.

4.3.7 Faraday's Laws of electro-magnetic induction

Statement

1. When ever there is a change in magnetic flux linked with the circuit an **emf** (e) is induced and is equal to rate of change of magnetic flux.
2. The emf induced is in such a direction that it apposes the cause.

Mathematically the induced emf is given by

$$e = -\frac{d\phi}{dt} \tag{4.29}$$

Here ϕ is magnetic flux linked with the circuit. For a coil of N turns the induced emf due to rate of change of flux is given by

$$e = -N\frac{d\phi}{dt} \tag{4.30}$$

Faraday's law in integral and differential forms

For a conducting loop linked with change in magnetic flux the rate of change flux is

$$\frac{d\phi}{dt} = \int_s \frac{\partial \vec{B}}{\partial t}.\vec{dS} \tag{4.31}$$

The induced emf in the the circuit is given by

$$e = \oint \vec{E}.\vec{dL} \tag{4.32}$$

Substituting the above in the equation 4.29 we get

$$\oint \vec{E}.\vec{dL} = -\int_s \frac{\partial \vec{B}}{\partial t}.\vec{dS} \tag{4.33}$$

Using the Stokes' theorem

$$\oint \vec{E}.\vec{dL} = \int_s (\nabla \times \vec{E}).\vec{dS} \tag{4.34}$$

and hence we can write

$$\int_s (\nabla \times \vec{E}).\vec{dS} = -\int_s \frac{\partial \vec{B}}{\partial t}.\vec{dS} \tag{4.35}$$

Thus finally it reduces to

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{4.36}$$

Thus Faraday's law in differential (Point form) and one of the Maxwell's equations.

4.4 Equation of continuity

In all processes involving motion of charge carriers the net charge is always conserved and is called the law of conservation of charges.

Let us consider a volume V. Let the charges flow in to and out of the volume V. Then the equation for the law of conservation could be written in the integral form as

$$\oint_s \vec{J}.\vec{dS} = -\frac{\partial}{\partial t}\int_v \rho_v\ dV \tag{4.37}$$

ρ_v is the volume density of charge and $\vec{J} = Ne\vec{v} = \rho_v\vec{v}$ is the current density.The negative sign indicates that the current density is due to the decrease in positive charge density inside the volume. Using the Gauss divergence theorem we can write

$$\oint_s \vec{J}.\vec{dS} = \oint_v (\nabla.\vec{J}).dV$$

Thus the equation 4.37 could be written as

$$\oint_v (\nabla.\vec{J}).dV = -\frac{\partial}{\partial t}\int_v \rho_v\ dV$$

The above equation could be reduced to

$$\oint_v (\nabla.\vec{J}).dV = -\int_v \frac{\partial \rho_v}{\partial t}\ dV$$

Thus the equation of continuity could be written as

$$\nabla.\vec{J} = -\frac{\partial \rho_v}{\partial t} \tag{4.38}$$

4.38 is also the law of conservation of charges.

Discussion on equation of continuity :

In case of DC circuits for steady currents the inward flow of charges is equal to the outward flow through a closed surface and hence $\frac{\partial \rho_v}{\partial t} = 0$. Thus the equation of continuity becomes $\nabla.\vec{J} = 0$.

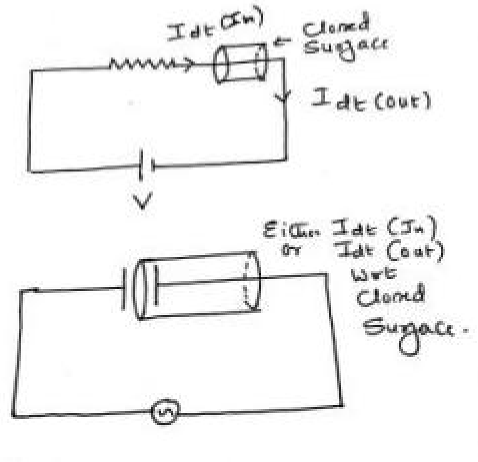

Figure 4.12: DC and AC circuits - Continuity equation

In case of AC circuits containing capacitors the equation $\nabla.\vec{J} = 0$ fails as follows. During the positive half cycle, say, the capacitor charges. If we imagine a closed surface enclosing the capacitor plate and the attached conductor there will be inward flow to the closed surface but not outward flow. Thus in order to rescue the equation of continuity Maxwell introduced the concept of displacement current density.

4.5 Displacement Current

4.5.1 Definition

Displacement current density is a correction factor introduced by Maxwell in order to explain the continuity of electric current in time-varying circuits. It has the same unit as electric current density. Displacement current is associated with magnetic current but it does not describe the flow of charge.

4.5.2 Maxwell-Ampere Law

Introducing the concept of displacement current for time varying circuits, Maxwell suggested corrections to the Amperes law. According to Gauss' Law

$$\nabla.\vec{D} = \rho_v$$

Differentiating the above equation with respect to time

$$\frac{\partial}{\partial t}\left(\nabla.\vec{D}\right) = \frac{\partial \rho_v}{\partial t}$$

$$\nabla.\frac{\partial \vec{D}}{\partial t} = \frac{\partial \rho_v}{\partial t} \tag{4.39}$$

The equation of continuity is given by

$$\nabla.\vec{J} = -\frac{\partial \rho_v}{\partial t}$$

Hence equation 4.39 could be written as

$$\nabla.\vec{J} = -\nabla.\left(\frac{\partial \vec{D}}{\partial t}\right)$$

$$\nabla.\left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) = 0$$

Hence for time varying circuits $\nabla.\vec{J} = 0$ does not hold good and instead $\nabla.\left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) = 0$ has to be used. Also $\vec{J}$ in Amperes Circuital law $\nabla \times \vec{H} = \vec{J}$ has to be replace with $\left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)$ Thus the Maxwell-Ampere law is given by

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{4.40}$$

In the above equation $\frac{\partial \vec{D}}{\partial t}$ is called displacement current.

4.5.3 Expression for Displacement current

Consider an AC circuit containing a capacitor as shown in the figure 4.13

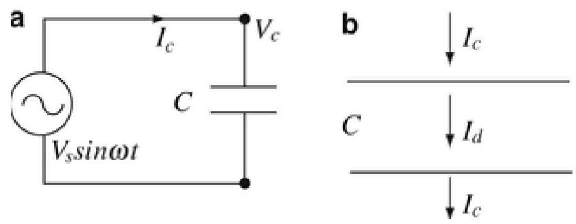

Figure 4.13: Displacement current

The displacement current in terms of displacement current density is given by

$$I_D = \left(\frac{\partial \vec{D}}{\partial t}\right).A \tag{4.41}$$

Here A is the area of the capacitor plates.The elecric flux density D is give by

$$D = \epsilon E \tag{4.42}$$

Here E is the electric field strength which is given by

$$E = \frac{V}{d} \tag{4.43}$$

Here d is the separation between the capacitor plates. V the applied potential is given by

$$V = V_s\ e^{jwt} \tag{4.44}$$

Using equations 4.42, 4.43 and 4.44 we get

$$D = \frac{\epsilon}{d} V_s\ e^{j\omega t} \tag{4.45}$$

Substituting for D in equation 4.41 from equation 4.45, we get

$$I_D = \frac{\partial}{\partial t}\left(\frac{\epsilon}{d}\ V_s\ e^{j\omega t}\right).A$$

Executing differentiation the displacement current is given by

$$I_D = \frac{j\omega\epsilon A}{d}\ V_s\ e^{jwt} \tag{4.46}$$

4.6 Maxwell's Equations

Using the laws and theorems discussed in this chapter Four Maxwell's equations for time-varying fields could be written as

1. Gauss' Law of Electrostatics $\nabla.\vec{D} = \rho_v$
2. Faraday's Law $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
3. Gauss' Law of Magnetic fields $\nabla.\vec{B} = 0$
4. Maxwell - Ampere Law $\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

The Four Maxwell's equations for static fields could be written as

1. $\nabla.\vec{D} = \rho_v$
2. $\nabla \times \vec{E} = 0$
3. $\nabla.\vec{B} = 0$
4. $\nabla \times \vec{H} = \vec{J}$

The above equations are used to study the electromagnetic waves.

Chapter 5

Electromagnetic waves

5.1 Introduction

The existence of EM waves was predicted by Maxwell theoretically using the point form of Faraday's Law of electromagnetic induction. As per Faraday's law a time varying magnetic field induces electric field which varies with respect to space and time. The reverse is also evident from the equations. Thus Electromagnetic wave is the propagation of energy in terms of varying electric and magnetic fields which are in mutually perpendicular directions and perpendicular to the direction of propagation.

5.2 Wave equation for EM waves in vacuum in terms of electric field using Maxwell's Equations

Consider the Maxwell's equations

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{5.1}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{5.2}$$

Substituting $D = \epsilon E$ and $B = \mu H$ in the above equations we get

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \tag{5.3}$$

$$\nabla \times \vec{H} = \vec{J} + \epsilon \frac{\partial \vec{E}}{\partial t} \tag{5.4}$$

To derive wave equation in terms of electric field, the term $\vec{H}$ has to be eliminated. Taking curl on both sides the equation 5.3 we get

$$\nabla \times \nabla \times \vec{E} = -\mu \frac{\partial}{\partial t}\left(\nabla \times \vec{H}\right) \tag{5.5}$$

According to vector analysis $A \times (B \times C) = B(A.C) - C(A.B)$. Thus

$$\nabla \times \nabla \times \vec{E} = \nabla \left(\nabla . \vec{E}\right) - \nabla^2 \vec{E}$$

As per the Maxwells equation $\nabla . \vec{D} = \rho_v$. Since $D = \epsilon E$ it could be written as $\nabla . \vec{E} = \frac{\rho_v}{\epsilon}$. Substituting in the above equation we get

$$\nabla \times \nabla \times \vec{E} = \nabla \left(\frac{\rho_v}{\epsilon}\right) - \nabla^2 \vec{E} \tag{5.6}$$

Substituting equation 5.6 in equation 5.5 we get

$$\nabla \left(\frac{\rho_v}{\epsilon}\right) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t}\left(\nabla \times \vec{H}\right) \tag{5.7}$$

Substituting equation 5.4 in 5.7 we have

$$\nabla \left(\frac{\rho_v}{\epsilon}\right) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t}\left(\vec{J} + \epsilon \frac{\partial \vec{E}}{\partial t}\right) \tag{5.8}$$

the above equation could be rewritten as

$$\nabla^2 \vec{E} - \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = \mu \frac{\partial \vec{j}}{\partial t} + \nabla \left(\frac{\rho_v}{\epsilon}\right) \tag{5.9}$$

The LHS in equation 5.9 represents a propagating wave and the RHS the source of origin of the wave. Here μ and ϵ are respectively Absolute permeability and Absolute permittivity of isotropic homogeneous medium. In case of propagation of EM wave in free space ($\vec{J} = 0, \rho_v = 0$) equation 5.9 reduces to

$$\nabla^2 \vec{E} - \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \tag{5.10}$$

Hence the electromagnetic wave equation in free space. Comparing the above equation with the general wave equation we get the velocity of the EM wave

$$\frac{1}{v^2} = \mu\epsilon \tag{5.11}$$

hence velocity of the EM wave

$$v = \frac{1}{\sqrt{\mu\epsilon}} \tag{5.12}$$

The velocity of propagation of EM Wave in vacuum

$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}} \approx 3 \times 10^8 ms^{-1} \tag{5.13}$$

5.3 Plane electromagnetic waves in vacuum

Electromagnetic waves that travels in one direction and uniform in the other two orthogonal directions is called plane electromagnetic waves. For example consider a plane electromagnetic wave traveling along z axis the electric and magnetic vibrations are uniform and confined to x-y plane.

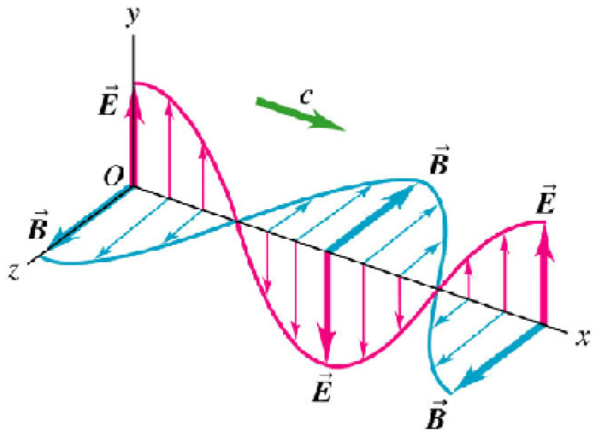

Figure 5.1: Plane Electromagnetic Waves

Consider a plane electromagnetic wave propagating along +ve x-axis. If the time varying electric and magnetic fields are along y and z axes respectively then we can write

$$\vec{E} = A\ Cos\left[\frac{2\pi}{\lambda}(x - ct)\right]\hat{i} \tag{5.14}$$

$$\vec{B} = \frac{1}{c}A\ Cos\left[\frac{2\pi}{\lambda}(x - ct)\right]\hat{j} \tag{5.15}$$

The ratio of the amplitudes of Electric and Magnetic fields from equations 5.14 and 5.15 is given by

$$\frac{E_y}{B_z} = c \tag{5.16}$$

Here 'c' is the velocity of light.

5.4 Polarization of Elecromagnetic waves

5.4.1 Transverse nature of electromagnetic waves

The electric and magnetic variations are mutually perpendicular and perpendicular to the direction of propagation. Thus electromagnetic waves are transverse in nature. Electromagnetic waves also exhibit polarization. Consider an electromagnetic wave propagating along z-axis. The the electric field vector of this electromagnetic wave makes an angle $theta$ with respect to x-axis, say. This electric vector could be resolved into two perpendicular components $\vec{E_x}$ and $\vec{E_y}$ along x and y axes respectively. Based on the magnitudes of the components and the phase difference between the components there are three kinds of polarization of electromagnetic waves. They are

1. Linearly Polarized EM waves
2. Circularly Polarized EM waves
3. Electrically Polarized EM waves

Linear polarization In case of linear polarization the amplitudes of $\vec{E_x}$ and $\vec{E_y}$ may or may not be equal and they are in phase(in unison). Thus the projection of the resultant $\vec{E}$ on a plane (x-y plane) perpendicular to the direction of propagation is a straight line. Thus linear polarization.

Circular polarization In case of circular polarization the amplitudes of $\vec{E_x}$ and $\vec{E_y}$ are equal in magnitude and the phase difference is 90°. Thus the projection of the resultant traces a circle on the plane perpendicular to the direction of propagation. Thus Circular polarization.

Elliptical polarization In case of circular polarization the amplitudes of $\vec{E_x}$ and $\vec{E_y}$ are unequal in magnitude and the phase difference is 90°. Thus the projection of the resultant traces an ellipse on the plane perpendicular to the direction of propagation. Thus Circular polarization.

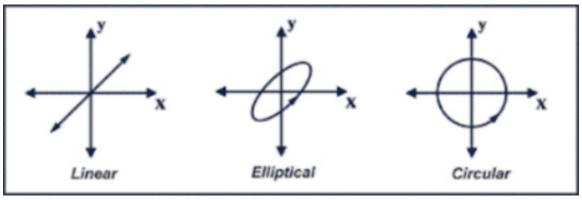

Figure 5.2: Polarization of Electromagnetic Waves

The linear, circular and elliptical polarization are as shown in the figure 5.2.